Problem 6C,7
Show that l^1 with norm defined by $\|(a_1, a_2, \ldots)\| = \sup_k |a_k|$ is not a Banach space.

Proof. Just take $l_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots)$. The limit should be $(\frac{1}{k})_{k=1,2,\ldots}$ but it is not in l^1. □

Problem 6C,8
Show that l^1 with norm defined by $\|(a_1, a_2, \ldots)\| = \sum_k |a_k|$ is a Banach space.

Proof. Let $l_n \in l^1$ be a Cauchy sequence and we need to find a limit in l^1.

- First we use Cauchy sequence to find a subsequence $\{l_{n_k}\}$ such that $\|l_{n_{k+1}} - l_{n_k}\| \leq \frac{1}{k^2}$. This is a standard procedure. First we find a N_1 such that for any $n, m \geq N_1$, $\|l_n - l_m\| \leq 1$, and then we just take $l_{n_1} = l_{N_1}$. Then we can find $N_2 > N_1$ such that for any $n, m \geq N_1$, $\|l_n - l_m\| \leq \frac{1}{2^2}$,and then take $l_{n_2} = l_{N_2}$. Repeating this process, we can find such subsequence.
- Take $a_k = l_{n_{k+1}} - l_{n_k}, k \geq 1; a_0 = l_{n_1}$. Then we know that $\sum_{k=0}^{N} a_k = l_{n_{N+1}}; and \sum_{k=0}^{N} |a_k| < 1000$ for any positive integer k. Now we can define $l_\infty = \sum_{k=0}^{\infty} a_k$ which is well-difined since for each entry, the limit do exist. And also we know that $\lim_{k\to\infty} l_{n_k} = l_\infty, l_\infty \in l^1$.
- By the Cauchy sequence of $l_n \in l^1$, we know in fact that $\lim_{k\to\infty} l_k = l_\infty$. This completes the proof.

□

Problem 6C,9
Show that the vector space $\mathcal{C}[0,1]$ with the norm defined by $\|f\| = \int_{[0,1]} |f| dx$ is not a Banach space.

Proof. For $k > 2$,Just take the function $f_k = 1, if\ x \in [0, \frac{1}{2} - \frac{1}{k}]; f_k = 0, if\ x \in [\frac{1}{2} + \frac{1}{k}, 1]$ and let f_k be linear in $[\frac{1}{2} - \frac{1}{k}, \frac{1}{2} + \frac{1}{k}]$. This is a Cauchy sequence but the limit should be $f = \chi_{[0,\frac{1}{2}]}(x)$ which cannot be modified to be a continuous function up to a null set. □

Problem 6C,15
Suppose V is a normed vector space and U is a subspace. Define $\|.\|$ on the quotient space V/U by

$$\|f + U\| = \inf\{\|f + g\| | g \in U\}$$

- Prove that $\|.\|$ is a norm on V/U if and only if U is a closed subspace of V.
- Prove that if V is a Banach space and U is a closed subspace, then the quotient space V/U with the norm defined above is also a Banach space.
- Prove that if U and V/U both are Banach spaces, then V is also Banach space.

Proof.

- The triangle inequality and homogeneity is quite obvious from the definition, which do not need U to be closed. Now we check the third property in the definition of norm. If U is a closed subspace, take $f \in V$ such that $\|f + U\| = 0$, then we can find $g_i \in U, i = 1, 2, \ldots$ such that $\|f + g_i\| \to 0$. Then we know that $\{g_i\}$ is Cauchy sequence in U. By closedness of U, we can find $g \in U$ which is the limt of $\{g_i\}$ and also $f + g = 0$, which implies $f \in U$ so that $[f + U] = [0]$ in the quotient space. Thus we know that $\|.\|$ is a norm on V/U. Conversely, take f_n be a Cauchy sequence in U and converge to f in V. Then one can see $\|f\| = 0$. Since it is norm, we know that $f \in U$ and thus U is a closed subsapce.
- Take a Cauchy sequence $[f_i + U]$ in V/U. Note first for any class $[x + U]$, we can always choose some represent element x such that $\|x'\| \leq 2\|x\|$. First we can choose a subsequence $[f_{n_k} + U]$ such that $\|f_{n_{k+1}} - f_{n_k} + U\| \leq \frac{1}{2^{K+1}}$. So we can find some g_{n_k} such that $\|f_{n_{k+1}} - f_{n_k} - g_{n_k}\| \leq \frac{1}{2^k}$. Now set $f'_{n_1} = f_{n_1}, f'_{n_2} = f_{n_2} - g_{n_1}, f'_{n_3} = f_{n_3} + g_{n_1} - g_{n_2}, \ldots$, so $[f'_{n_k} + U] = [f_{n_k} + U], \|f'_{n_{k+1}} - f'_{n_k}\| \leq \frac{1}{2^k}$. Since V is a Banach space, we know that these exists $f_\infty = \lim_{k\to\infty} f'_{n_k}$. And since obviosly for any $x \in V, \|x\| \geq \|x + U\|$, we know that $[f_\infty + U] = \lim_{k\to\infty}[f'_{n_k} + U] = \lim_{k\to\infty}[f_{n_k} + U]$. Thus $[f_\infty + U] = \lim_{k\to\infty}[f_k + U]$. (**Remark:**We can also use 6.41 to prove this.)

- Take any Cauchy sequence f_n in V. Since for any $x \in V, \|x\| \geq \|x+U\|$, we know that $[f_n + U]$ is also Cauchy sequence in V/U. Then by assumption, we can have a limit $[f_\infty + U]$. Now we can choose $g_{n_k} \in U$ such that $\|f_{n_k} - f_\infty - g_{n_k}\| \leq 2\|f_{n_k} - f_\infty + U\|$. Recall that f_{n_k} is Cauchy sequence, thus g_{n_k} is also Cauchy sequence sequence in U and we can get a limit g_∞. Thus $f_\infty + g_\infty = \lim_{k\to\infty} f_{n_k} = \lim_{k\to\infty} f_k$. □

Problem 6D,2
Suppose ϕ is a linear functional on a vector space V. Prove that if U is vector subspace of V and $\text{null}\phi \subset U$, then $U = null\phi$ or $U = V$.

Proof. Without loss of generality, we assume ϕ is not zero functional. Then we can find $a \in V$ such that $f(a) = 1$. Then for any $x \in V$, we know that

$$f(x - f(x)a) = f(x) - f(x) = 0$$

which imply $x - f(x)a \in null\phi$. In other words, $null\phi \oplus span\{a\} = V$. Then the result follows directly. □

Problem 6D,3
Suppose ϕ, ψ are linear functional on the same vector space. Prove that $null\phi \subset null\psi$ if and only if there exists some $\alpha \in \mathbb{F}$ such that $\psi = \alpha\phi$.

Proof. Let V be the vector space. By the previous exercise, we know that $null\psi = V$ or $null\psi = \phi$. In the first case, $\psi = 0$ so we can just take $\alpha = 0$. In the second case, take $a \in V$ such that $\phi(a) = 1$. Then we know that for any $x \in V, x - \phi(x)a \in null\phi$, so also $x - \phi(x)a \in null\psi$. So $\psi(x) = \psi(a)\phi(x)$. Then we can take $\alpha = \psi(a)$. □

Problem 6D,18
Suppose V is a normed vector space such that its dual $V^{'}$ is separable. Prove that V is separable.

Proof: Take $B = \{\phi \in V^{'} | \|\phi\| = 1\}$ which is just unit ball in $V^{'}$.

- We first show that B is separable. In fact, since $V^{'}$ is separable, we can find dense set $\{\phi_n\}_{n=1,2,\ldots}$ and without loss of generality $\phi_n \neq 0$ for all n. Take $\psi_n = \frac{\phi_n}{\|\phi\|}$. Then we can easily check that $\psi_k, k = 1, 2, \ldots$ is dense set in B. So we have proved that B is also separable.
- For any ψ_n, we can find $x_n \in V$ such that $\|x_n\| = 1, \psi_n(x_n) \geq \frac{1}{2}$. Now take $V_0 = \overline{span\{x_n\}_{n=1,2,\ldots}}$. Obvioully V_0 is separable since rational combination of x_n is countably dense subset of V_0.
- We claim that $V_0 = V$ and then the proof is complete. In fact, if there exists some $x \in V - V_0$, then from Hahn-Banach theorem, we know that we can find some $\phi_0 \in V^{'}$ such that $\phi_0|_{V_0} = 0, \|\phi_0\| = 1$ so $\phi_0 \in B$.Then we know that for any positive integer n,

  $$\|\psi_n - \phi_0\| = \sup_{\|x\|=1} |\psi_n(x) - \phi_0(x)| \geq |\psi_n(x_n) - \phi_0(x_n)| = |\psi_n(x_n)| \geq \frac{1}{2}$$

  This contradicts to the first step where we have proved that $\psi_k, k = 1, 2, \ldots$ is dense set in B. So in fact we do have $V_0 = V$.

Problem 6D,20
Define $\Phi : V \to V^{''}$ by

$$(\Phi f)(\phi) = \phi(f).$$

for $f \in V, \phi \in V^{'}$. Show that $\|\Phi(f)\| = \|f\|$.

Proof: By definition, for $f \in V$

$$\|\Phi(f)\| = \sup_{\|\phi\|=1} (\Phi f)(\phi) = \sup_{\|\phi\|=1} \phi(f) \leq \|f\|.$$

On the other hand, from Hahn-Bnanch theorem, we know that for $f \in V$, there exists some ϕ_0 such that $\phi_0(f) = \|f\|, \|\phi_0\| = 1$. From this fact, we know that

$$\sup_{\|\phi\|=1} \phi(f) \geq \|f\|.$$

combining these, we get $\|\Phi(f)\| = \|f\|$.